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Eisenstein series and periods

Thesis BSc Mathematics

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Abstract

In recent years, Zudilin and Rogers have developed a method to write L-values attached to elliptic curves as periods. In order to apply this to a broader collection of L-values, we define Eisenstein series and determine their Fourier series at the cusps. As an illustrating example, we write the L-values of an elliptic curve of conductor 32 as an integral of Eisenstein series and evaluate the value at 4 explicitly as a period.

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1 Introduction

1.1 Periods

At the start of a mathematics study, students learn successively about different kind of numbers. We start by learning about natural numbers

$$\mathbb{N} = \{0, 1, 2, 3, \dots\}.$$

Afterwards we add the negative numbers to get the integers:

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}.$$

Then by including fractions we get the rationals:

$$\mathbb{Q} = \left\{ \frac{p}{q} \;\middle|\; p, q \in \mathbb{Z},\ q \neq 0 \right\}.$$

By adding the limits of Cauchy sequences we get the real numbers $\mathbb{R}$. By formally adding an element i whose square is -1 and extending it linearly we obtain the complex numbers:

$$\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}.$$

An useful property of the complex numbers is that they form an algebraically closed field, which means that any nonconstant polynomial with complex coefficients has a root in $\mathbb{C}$. If we restrict to the numbers that are the solutions of polynomial (or equivalently, algebraic) equations with rational coefficients, we obtain the algebraic numbers $\overline{\mathbb{Q}} \subset \mathbb{C}$.

We therefore have the hierarchy

$$\begin{array}{ccccccc} \mathbb{N} & \subset & \mathbb{Z} & \subset & \mathbb{Q} & \subset & \overline{\mathbb{Q}} \\ & & & & \cap & & \cap. \\ & & & & \mathbb{R} & \subset & \mathbb{C} \end{array}$$

Often numbers are classified by considering their position in this hierarchy. There is a big difference in size of the set of algebraic numbers $\overline{\mathbb{Q}}$ compared to the set of complex numbers $\mathbb{C}$. The set of algebraic numbers is countable, as the set of polynomials can be enumerated, the complex numbers on the other hand are uncountable, by Cantor's diagonal argument. Due to the small size of the collection of algebraic numbers, many important constants, such as π and $\log(2)$ are not contained in it (Lindemann, 1882).

However, it is possible to define a class of numbers that includes these numbers, and much more, while still remaining countable. A natural choice for such a class are the periods. A period is a complex number which real part and imaginary part are both (absolutely convergent) integrals of rational functions with rational coefficients over domains in $\mathbb{R}^n$ defined by polynomial inequalities with rational coefficients [2].

The functions can also be chosen to be algebraic instead without any harm. The set of periods $\mathscr{P}$ forms a countable ring, which contains the algebraic numbers $\overline{\mathbb{Q}}$. For example,

$$\sqrt{2} = \int_{2x^2 \le 1} \mathrm{d}x.$$

The ring also contains other important constants such as

$$\pi = \iint_{x^2+y^2 \le 1} \mathrm{d}x\,\mathrm{d}y = \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1+x^2}$$

and integer values of the Riemann zeta function, $k > 1$,

$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k} = \underset{[0,1]^k}{\int \cdots \int} \frac{\mathrm{d}x_1\,\mathrm{d}x_2 \ldots \mathrm{d}x_k}{1 - x_1 x_2 \ldots x_k}.$$

The extended period ring $\widehat{\mathscr{P}} := \mathscr{P}[1/\pi]$ contains a large collection of natural examples, such as values of generalized hypergeometric functions at algebraic points [6] and special L-values. As an example, a theorem by Beilinson and Deninger–Scholl states that the (non-critical) value of the L-series attached to a cusp form $f(\tau)$ of weight k at a positive integer $m \ge k$ (see the definitions and formula (1) below) belongs to $\widehat{\mathscr{P}}$. Although the proof of the theorem is effective, computing these L-values as periods remains a very tough problem even in particular cases. Most of these computations are motivated by (conjectural) evaluations of the logarithmic Mahler measures of multivariate polynomials as L-values, where a Mahler measure of an polynomial $P \in \mathbb{Z}[x_1, \ldots, x_k]$ is defined as the following integral:

$$m(P) = \frac{1}{(2\pi i)^k} \underset{|x_1| = \cdots = |x_k| = 1}{\int \cdots \int} \log|P(x_1, \ldots, x_k)| \frac{\mathrm{d}x_1}{x_1} \cdots \frac{\mathrm{d}x_k}{x_k}.$$

With this purpose, Rogers and Zudilin have developed a setup [3, 4] for writing L-values $L(E, 2)$ of cusp forms $f(\tau)$ of weight 2 as periods. Zudilin [9] later described an algorithm behind the method, which is not restricted to the weight. This can be used in principle to compute arbitrary $L(E, k)$-values of an elliptic curve as periods, provided $k \ge 2$. However, even with this method, evaluating an L-value as a period remains a difficult task.

1.2 Modular forms and L-functions

In this thesis, a general notion of Eisenstein series is introduced and their Fourier series at cusps are studied. They are used to help with executing the method of Rogers and Zudilin [3, 4]. We present an example of evaluating $L(E, 4)$ as a period, a task that was never explicitly one before. After this, we show how the results on Eisenstein series can be used to write the L-values of the same curve as an integral of products of two of such series. The main results in this thesis are Theorems 1 (section 2), 2 (section 7) and 3 (section 8).

Two important concepts used in this thesis are modular forms and modular functions, which we shall now define. Let Γ be a subgroup, of finite index, of the modular group

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, ad - bc = 1 \right\},$$

which acts on the extended upper half plane (with added cusps)

$$\overline{\mathbb{H}} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\} \cup \mathbb{Q} \cup \{i\infty\}$$

by Möbius transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d}.$$

A modular form f of weight k for Γ is a holomorphic function on $\overline{\mathbb{H}}$ such that

$$f\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\tau\right) = (c\tau + d)^k f(\tau) \text{ for all } \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma,$$

along with the requirement that $f(\tau - \frac{c}{d})$ possesses a Fourier expansion $\sum_{n=0}^{\infty} a_n e^{2\imath i n\tau/N}$ (for some $N \in \mathbb{N}$) for every $\frac{c}{d} \in \mathbb{Q}$. A cusp form is a modular form such that $f(\tau)$ vanishes at $\tau \in \mathbb{Q} \cup \{i\infty\}$, that is, at every cusp. A modular function f is a meromorphic function that is Γ-invariant (it has weight $k = 0$) along with the property that $f(\tau - \frac{c}{d})$ possesses a Fourier expansion $\sum_{n=-m}^{\infty} a_n e^{2\imath i n\tau/N}$ (for some $N, m \in \mathbb{N}$) for every $\frac{c}{d} \in \mathbb{Q}$.

Throughout this thesis, we will use the notation $q = e^{2\pi i\tau}$ for τ in the upper half plane $\mathrm{Im}\,\tau > 0$, so $|q| < 1$. With this notation, modular forms are therefore power series in $q^{1/N}$, for some natural number N, while modular functions are Laurent series in $q^{1/N}$ with finitely many negative powers. For functions of variable q or τ, we will use the differential operator

$$\delta = \frac{1}{2\pi i}\frac{\mathrm{d}}{\mathrm{d}\tau} = q\frac{\mathrm{d}}{\mathrm{d}q}$$

and denote by δ^{-1} the corresponding anti-derivative normalized by 0 at $\tau = i\infty$ (or at $q = 0$):

$$\delta^{-1} f = \int_0^q f \frac{\mathrm{d}q}{q}.$$

In particular, for a modular form $f(\tau) = \sum_{n=1}^{\infty} a_n q^n$ whose expansion vanishes at infinity, we have

$$L(f, k) := \frac{1}{(k-1)!}\int_0^1 f \log^{k-1} q \frac{\mathrm{d}q}{q} = \frac{(2\pi)^k}{(k-1)!}\int_0^{\infty} f(it) t^{k-1}\,\mathrm{d}t, \qquad (1)$$

and

$$L(f,k) = \sum_{n=1}^{\infty} \frac{a_n}{n^k} = (\delta^{-k} f)|_{q=1}$$

whenever the latter sum makes sense.

We use two standard constructors of modular forms and modular functions: Eisenstein series and Dedekind's eta function. The Eisenstein series will be the focus of the first part of the thesis; they are defined in the next section. The Dedekind eta function is defined as follows:

$$\eta(\tau) := q^{1/24} \prod_{m=1}^{\infty} (1-q^m) = \sum_{n=-\infty}^{\infty} (-1)^n q^{(6n+1)^2/24};$$

its modular involution reads

$$\eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau). \tag{2}$$

We also set $\eta_k(\tau) := \eta(k\tau)$ for short.

As an illustration of the ideas in this paper, the L-values of an elliptic curve of conductor 32 will be studied. An example of such a curve is

$$y^2 = x^3 - x.$$

By the modularity theorem for any elliptic curve E, there exists an associated cusp form f of weight 2 such that the L-function $L(E,s)$ of the curve coincides with the L-function $L(f,s)$. In the case of conductor 32, this cusp form is

$$f(\tau) = \eta_4^2 \eta_8^2.$$

This special modular form satisfies many properties, such as multiplicativity of the Fourier coefficients: if

$$f(\tau) = \sum_{n=1}^{\infty} a_n q^n,$$

then $a_{nm} = a_n a_m$ for n and m relatively prime. This cusp form is intimately connected to the number of points N_p on the curve modulo a prime p (including the point at infinity), as its Fourier coefficients satisfy $a_p = p + 1 - N_p$ for almost all primes p.

We will also sometimes use the generalized hypergeometric functions, which is defined by the series

$${}_{k+1}F_k\left(\begin{matrix} a_0,\, a_1, \dots, a_k \\ b_1, \dots, b_k \end{matrix} \,\middle|\, z\right) = \sum_{n=0}^{\infty} \frac{(a_0)_n (a_1)_n \dots (a_k)_n}{(b_1)_n \dots (b_k)_n} \frac{z^n}{n!}$$

in the disk $|z| < 1$; here $(a)_n := \Gamma(a+n)/\Gamma(a) = \prod_{m=0}^{n-1}(a+m)$ represents the Pochhammer symbol. The properties of the series, such as integral representations and analytic continuation can be found in Slater's treatise [6, §4].

In later sections, sometimes an equality is established between two different modular forms of weight k for some subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ that has finite index m using only the first terms of the Fourier expansions. This is justified by using the Sturm bound [7, Corollary 9.20], which states that if the first $km/12$ terms of the Fourier expansions are equal, then the forms are identical. In practice, the constants k (the weight) and m (the level) tend to be small, so equality is easy to verify.

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2 Eisenstein series

The work of Y. Yang [8] provided us with transformation laws for the generalized Dedekind eta functions of level N:

$$g_{a,b}(\tau) = q^{B(a/N)/2} \prod_{\substack{m\geq 1 \\ m\equiv a \bmod N}} (1-\zeta_N^b q^m) \prod_{\substack{m\geq 1 \\ m\equiv -a \bmod N}} (1-\zeta_N^{-b} q^m), \tag{3}$$

$$\text{where } q = e^{2\pi i\tau} \text{ and } B(x) = \{x\}^2 - \{x\} + 1/6$$

whose logarithms may be viewed as Eisenstein series of weight zero. Similarly, there are very classical transformation laws for the following Eisenstein series of weight k and level N, as described by Schoeneberg [5, Chapter 7], if $k > 2$:

$$G_{N,k,(a,b)}(\tau) = \sum_{\substack{m\equiv a \\ n\equiv b}\ \bmod N}{}' (m\tau+n)^{-k}, \tag{4}$$

where the dash means that $(m,n) = (0,0)$ is excluded from summation. When $k = 1$ or 2, it is defined similarly, though analytic continuation is required to circumvent the lack of absolute convergence.

We will unify these two notions and use the transformation laws to find two different expansions of the series at their cusps.

In order to do this, we take integers N, k, a and b, where k is nonnegative and N is positive. Before introducing general Eisenstein series, we need to define their constant terms:

$$\gamma_{a,b}(\tau) = \begin{cases} \beta_k \alpha_{a,b}^{N,k} & \text{if } k \neq 0, 2 \\ -\pi i \tau N B(a/N) & \text{if } k = 0 \\ \beta_2 \big(\alpha_{a,b}^{N,2} - \frac{2\pi i}{N^2(N\tau - N\bar{\tau})}\big) & \text{if } k = 2; \end{cases}$$

where

$$\beta_k = \frac{(k-1)!}{(-2\pi i)^k},$$

and $\alpha = \alpha_{a,b}^{N,k}$ is defined by setting

$$\alpha_{a,b}^{N,k} = 0 \text{ if } a \equiv 0 \bmod N$$

and otherwise, when $k > 1$:

$$\alpha_{a,b}^{N,k} = \sum_{\substack{m \in \mathbb{Z} \\ m \equiv b}}{}' m^{-k} = \frac{1}{N^k}\left[\zeta\left(k, \left\{\frac{b}{N}\right\}\right) + (-1)^k \zeta\left(k, -\left\{\frac{b}{N}\right\}\right)\right],$$

and when $k = 1$:

$$\begin{aligned} \alpha_{a,b}^{N,k} =& \frac{1}{N}\lim_{s\to 0}\left[\zeta\left(1+s, \left\{\frac{b}{N}\right\}\right) - \zeta\left(1+s, -\left\{\frac{b}{N}\right\}\right)\right] \\ &- \frac{\pi i}{N}\left[\zeta\left(0, \left\{\frac{a}{N}\right\}\right) - \zeta\left(0, -\left\{\frac{a}{N}\right\}\right)\right], \end{aligned}$$

where $\zeta(s,t)$ denotes the Hurwitz zeta function, and $\{x\}$ denotes the fractional part of x. Now we define the Eisenstein series $E_{a,b} = E_{a,b}^{N,k}$ of level N and weight k as

$$E_{a,b}(\tau) = \gamma_{a,b}(\tau) + \sum_{\substack{n,m \geq 1 \\ m \equiv a \bmod N}} \zeta_N^{bn} n^{k-1} q^{mn} + (-1)^k \sum_{\substack{n,m \geq 1 \\ m \equiv -a \bmod N}} \zeta_N^{-bn} n^{k-1} q^{mn}, \qquad (5)$$

where $\zeta_N = e^{\frac{2\pi i}{N}}$. Note that we can even write $\gamma_{a,b} = \gamma_{*,b}$ if $k > 1$ and $a \not\equiv 0$, because $\gamma_{a,b}$ does not depend on a in that case.

With this definition we will later see that

$$E_{a,b}^{N,0}(\tau) = -\log g_{a,b}(\tau), \qquad (6)$$

and

$$E_{a,b}^{N,k}(\tau) = \beta_k G_{N,k,(a,b)}(N\tau) \text{ for any } k > 0. \qquad (7)$$

We are now ready to formulate the theorem.

3 Expansions for the Eisenstein series at its cusps

In order to express the expansions of $E_{a,b}$ around $\frac{c}{N}$ where c is an integer, define

$$E^{N,k}_{a,b,c}(\tau) = E_{a,b,c}(\tau) = E_{a,b}\left(\frac{c}{N} + \tau\right).$$

The following theorem gives two Fourier expansions for this general Eisenstein series.

Theorem 1. *For a, b, c arbitrary integers, $E_{a,b,c} = E^{N,k}_{a,b,c}$ possesses the following expansions:*

$$E_{a,b,c}(\tau) = E_{a,-a'}(\tau) + \gamma_{a,b}(\tau) - \gamma_{a,-a'}(\tau) + \delta_{k,0} \cdot \pi i c B(a/N) \tag{8}$$

and

$$\begin{aligned} E_{a,b,c}(\tau)(N\tau)^k = E_{a',a}\left(\frac{-1}{N^2\tau}\right) + \gamma_{a',b'}\left(\frac{-1}{N^2\tau}\right) - \gamma_{a',a}\left(\frac{-1}{N^2\tau}\right) \\ + \delta_{k,0} \cdot \pi i\big(\mu - icB(a'/N)\big), \end{aligned} \tag{9}$$

where δ_{ij} denotes the Kronecker delta, $a' = -ac - b$, $b' = a(c^2+1) + bc$ and μ is a rational number such that

$$\mu \equiv -\frac{((a')^2c + 2a'b')(c^2+1) + (b')^2c}{N^2} + \frac{a'(c^2+1) + b'(c+1)}{N} - \frac{1}{2} \bmod 2. \tag{10}$$

If we take c to be zero, we obtain the following corollary.

Corollary 1. *For any integers a and b, $E_{a,b} = E^{N,k}_{a,b}$ satisfies*

$$E_{a,b}(\tau)(N\tau)^k = E_{-b,a}\left(\frac{-1}{N^2\tau}\right) + \delta_{k,0}\pi i\mu \tag{11}$$

with

$$\mu \equiv -\frac{2ab}{N^2} + \frac{a-b}{N} - \frac{1}{2} \bmod 2.$$

Below we make occasional use of the identity

$$E^{N,k}_{-a,-b}(\tau) = (-1)^k E^{N,k}_{a,b}(\tau) + \delta_{k,0}\pi i\tau N(B(a/N) - B(-a/N)),$$

which is an immediate consequence of the definition of $E^{N,k}_{a,b}$.

4 Proof of Theorem 1 for weight equal to 0

Recall Yang's [8, Theorem 1] definition of the generalized Dedekind eta functions $\eta_{a,b}(\tau)$ of level N with $q = e^{2\pi i\tau}$:

$$g_{a,b}(\tau) = q^{NB(a/N)/2} \prod_{\substack{m\geq 1 \\ m\equiv a \bmod N}} (1 - \zeta_N^b q^m) \prod_{\substack{m\geq 1 \\ m\equiv -a \bmod N}} (1 - \zeta_N^{-b} q^m)$$

with

$$B(x) = \{x\}^2 - \{x\} + 1/6,$$

where the notation ζ_N represents $e^{2\pi i/N}$. (Note that our $g_{a,b}(\tau)$ coincide with $E_{a,b}(N\tau)$ in Yang's paper.)

Therefore we have

$$\begin{aligned} g_{a,b}(c/N+\tau) &= \exp(\pi i(c+N\tau)B(a/N)) \\ &\quad\times \prod_{\substack{m\geq 1\\ m\equiv a \bmod N}} (1-\zeta_N^b \exp(2\pi i m(c/N+\tau))) \\ &\quad\times \prod_{\substack{m\geq 1\\ m\equiv -a \bmod N}} (1-\zeta_N^{-b} \exp(2\pi i m(c/N+\tau))) \\ &= \exp(\pi i(c+N\tau)B(a/N)) \\ &\quad\times \prod_{\substack{m\geq 1\\ m\equiv a \bmod N}} (1-\zeta_N^{b+mc} q^m) \prod_{\substack{m\geq 1\\ m\equiv -a \bmod N}} (1-\zeta_N^{-b+mc} q^m). \end{aligned}$$

In what follows we fix the principal value of the logarithm, so that $-\log(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n}$ for z inside the unit disk. Taking logarithms on both sides we obtain

$$\begin{aligned} \log g_{a,b}(c/N+\tau) =& \pi i c B(a/N) + \pi i \tau N B(a/N) \\ & \sum_{\substack{m\geq 1\\ m\equiv a \bmod N}} \log(1-\zeta_N^{b+ac} q^m) \sum_{\substack{m\geq 1\\ m\equiv -a \bmod N}} \log(1-\zeta_N^{-b+ac} q^m) \\ =& \pi i c B(a/N) + \pi i \tau N B(a/N) \\ & - \sum_{\substack{m,n\geq 1\\ m\equiv a \bmod N}} \frac{\zeta_N^{bn+acn} q^{mn}}{n} - \sum_{\substack{m,n\geq 1\\ m\equiv -a \bmod N}} \frac{\zeta_N^{-bn-acn} q^{mn}}{n} \\ =& -E_{a,ac+b}(\tau) + \pi i c B(a/N), \end{aligned}$$

which establishes both the identity $-\log g_{a,b} = E_{a,b}$ in (6) and the Fourier expansion (8) in the theorem.

Now we will proceed with the expansion (9). For this we will apply [8, Theorem 1] on $g_{a,b}(\tau/N)$, where we choose the matrix γ to be

$$A = \begin{pmatrix} c & -c^2-1 \\ 1 & -c \end{pmatrix}. \tag{12}$$

That theorem implies

$$g_{a',b'}\Big(\frac{1}{N}\left(\begin{smallmatrix} c & -c^2-1 \\ 1 & -c \end{smallmatrix}\right)\tau\Big) = e^{\pi i \mu} g_{a,b}(\tau/N),$$

where

$$(a' \quad b') = (a \quad b)\begin{pmatrix} c & -c^2-1 \\ 1 & -c \end{pmatrix}^{-1} = -(a \quad b)\begin{pmatrix} c & -c^2-1 \\ 1 & -c \end{pmatrix} = (-ac-b \quad a(c^2+1)+bc). \tag{13}$$

Therefore,

$$g_{a,b}(\tau/N) = g_{a',b'}\left(\frac{c\tau-(c^2+1)}{N(\tau-c)}\right)e^{-\pi i\mu}$$

implying

$$\begin{aligned} g_{a,b}(\tau) &= g_{a',b'}\left(\frac{cN\tau-(c^2+1)}{N(N\tau-c)}\right)e^{-\pi i\mu} \\ &= g_{a',b'}\left(\frac{c\tau-(c^2+1)/N}{N\tau-c}\right)e^{-\pi i\mu}, \end{aligned}$$

By setting

$$\tau' = \left.\frac{ct-(c^2+1)/N}{Nt-c}\right|_{t=c/N+\tau} = \frac{c}{N}-\frac{1}{N^2\tau},$$

we have

$$\begin{aligned} \log g_{a',b'}(\tau') &= -E_{a',b'}\left(\frac{c}{N}-\frac{1}{N^2\tau}\right) \\ &= \pi icB(a'/N) - \frac{\pi iB(a'/N)}{N\tau} \\ &\quad - \sum_{\substack{m,n\geq 1 \\ m\equiv a' \bmod N}} \frac{\zeta_N^{b'n}}{n}\exp\left(\frac{2\pi ic}{N}-\frac{2\pi imn}{N^2\tau}\right) \\ &\quad - \sum_{\substack{m,n\geq 1 \\ m\equiv -a' \bmod N}} \frac{\zeta_N^{-b'n}}{n}\exp\left(\frac{2\pi ic}{N}-\frac{2\pi imn}{N^2\tau}\right) \\ &= \pi icB(a'/N) - \frac{\pi iB(a'/N)}{N\tau} \\ &\quad - \sum_{\substack{m,n\geq 1 \\ m\equiv a' \bmod N}} \frac{\zeta_N^{b'n+a'cn}}{n}\exp\left(\frac{-2\pi imn}{N^2\tau}\right) \\ &\quad - \sum_{\substack{m,n\geq 1 \\ m\equiv -a' \bmod N}} \frac{\zeta_N^{-b'n-a'cn}}{n}\exp\left(\frac{-2\pi imn}{N^2\tau}\right). \end{aligned}$$

Thus, we obtain

$$\log g_{a,b}\left(\frac{c}{N}+\tau\right) = -\pi i\mu + \pi i c B(a'/N) - \frac{\pi i B(a'/N)}{N\tau} - \sum_{\substack{m,n\geq 1\\ m\equiv a' \bmod N}} \frac{\zeta_N^{an}}{n}\exp\left(\frac{-2\pi mni}{N^2\tau}\right) - \sum_{\substack{m,n\geq 1\\ m\equiv -a' \bmod N}} \frac{\zeta_N^{-an}}{n}\exp\left(\frac{-2\pi mni}{N^2\tau}\right),$$

and the Fourier expansion (9) follows.

5 Proof of Theorem 1 for positive weight

In [5, Chapter 7], the following expansions for $G_{N,k,(a,b)}$ in (4) are given:

$$\begin{aligned} G_{N,k,(a,b)}(\tau) &= \alpha_{a,b}^{N,k} + \frac{1}{\beta_k}\sum_{m\equiv a \bmod N}\sum_{nm>0} n^{k-1}\cdot \operatorname{sgn} n \cdot e^{\frac{2\pi i}{N}(bn+\tau nm)} \\ &\quad - \delta_{k,2}\frac{2\pi i}{N^2(\tau-\overline{\tau})} \\ &= \alpha_{a,b}^{N,k} \\ &\quad + \frac{1}{\beta_k}\Bigg(\sum_{\substack{m,n\geq 1\\ m\equiv a \bmod N}} n^{k-1}e^{\frac{2\pi i}{N}(bn+\tau nm)} + (-1)^k \sum_{\substack{m,n\geq 1\\ m\equiv -a \bmod N}} n^{k-1}e^{\frac{2\pi i}{N}(-bn+\tau nm)}\Bigg) \\ &\quad - \delta_{k,2}\frac{2\pi i}{N^2(\tau-\overline{\tau})}. \end{aligned}$$

Thus $E_{a,b}(\tau) = G_{N,k,(a,b)}(N\tau)$ as previously asserted in (7).

We will now derive Fourier expansions of $E_{a,b,c}^{N,k}(\tau)$ for $k>0$, in terms of τ and $-\frac{1}{N^2\tau}$, similarly to what was done for $E_{a,b,c}^{N,0}$.

5.1 Expansion in τ

The Fourier expansion (8) is obtained simply by writing out the definition of $E_{a,b}$ if $k \neq 2$:

$$\begin{aligned}
E^{N,k}_{a,b,c}(\tau) &= \beta_k \alpha^{N,k}_{a,b} \\
&\quad + \Bigg(\sum_{\substack{m,n\geq 1 \\ m\equiv a \bmod N}} n^{k-1} e^{\frac{2\pi i}{N}(bn+(c+N\tau)nm)} + (-1)^k \sum_{\substack{m,n\geq 1 \\ m\equiv -a \bmod N}} n^{k-1} e^{\frac{2\pi i}{N}(-bn+(c+N\tau)nm)} \Bigg) \\
&= \beta_k \alpha^{N,k}_{a,b} \\
&\quad + \Bigg(\sum_{\substack{m,n\geq 1 \\ m\equiv a \bmod N}} n^{k-1} \zeta_N^{bn+cmn} q^{mn} + (-1)^k \sum_{\substack{m,n\geq 1 \\ m\equiv -a \bmod N}} n^{k-1} \zeta_N^{-bn+cmn} q^{mn} \Bigg) \\
&= \beta_k \alpha^{N,k}_{a,b} \\
&\quad + \Bigg(\sum_{\substack{m,n\geq 1 \\ m\equiv a \bmod N}} n^{k-1} \zeta_N^{bn+acn} q^{mn} + (-1)^k \sum_{\substack{m,n\geq 1 \\ m\equiv -a \bmod N}} n^{k-1} \zeta_N^{-bn-acn} q^{mn} \Bigg).
\end{aligned}$$

When $k = 2$ we have the same expression, with the extra term

$$-\beta_2 \frac{2\pi i}{N^2((c+N\tau) - (\overline{c+N\tau}))} = -\beta_2 \frac{2\pi i}{N^2(N\tau - N\overline{\tau})} = -\beta_2 \frac{\pi}{N^3 \operatorname{Im}(\tau)}$$

included. This establishes (8) for any $k > 0$.

5.2 Expansion in $\frac{-1}{N^2\tau}$

Now we will derive an Fourier expansion in terms of $\frac{1}{N^2\tau}$. For every $B = \left(\begin{smallmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{smallmatrix}\right) \in \Gamma$, we have

$$G_{N,k,(a,b)}(B\tau) = (b_{21}\tau + b_{22})^k G_{N,k,(a,b)B^t}(\tau)$$

(see [5, Chapter 7]). Recall (13), then

$$G_{N,k,(a',b')}\left(\frac{c\tau - (c^2+1)}{\tau - c}\right) = G_{N,k,(a',b')}(A\tau) = G_{N,k,(a,b)}(\tau)(\tau - c)^k.$$

Substituting $c + N\tau$ for τ, we obtain

$$\begin{aligned}
G_{N,k,(a,b)}(c+N\tau)(N\tau)^k &= G_{N,k,(a',b')}\left(\frac{c(c+N\tau) - (c^2+1)}{c+N\tau - c}\right) \\
&= G_{N,k,(a',b')}\left(\frac{cN\tau - 1}{N\tau}\right) \\
&= G_{N,k,(a',b')}\left(c - N\frac{1}{N^2\tau}\right)
\end{aligned}$$

meaning that

$$E_{a,b,c}(\tau)(N\tau)^k = E_{a',b',c}\Big(-\frac{1}{N^2\tau}\Big).$$

Using the first Fourier expansion (8) for $E_{a',b',c}$, we obtain

$$\begin{aligned} E^{N,k}_{a,b,c}(\tau)(N\tau)^k = \beta_k\alpha^{N,k}_{a,b} \\ &+ \sum_{\substack{m,n\geq 1\\ m\equiv a' \bmod N}} n^{k-1}\zeta_N^{b'n+a'cn}\exp\Big(-\frac{-2\pi mni}{N^2\tau}\Big) \\ &+ (-1)^k \sum_{\substack{m,n\geq 1\\ m\equiv -a' \bmod N}} n^{k-1}\zeta_N^{-b'n-a'cn}\exp\Big(\frac{-2\pi mni}{N^2\tau}\Big), \end{aligned}$$

which is precisely (9).

6 Expressing double series as Eisenstein series

More generally, we may consider series of the form

$$S(\tau) = \sum_{n,m\geq 1} n^{k-1} f(n)g(m)q^{mn},$$

with f and g both N-periodic and satisfying the parity constraint

$$f(-a)g(-b) = (-1)^k f(a)g(b) \text{ for all integers } a, b.$$

One advantage of the Eisenstein series introduced here is that it allows us to represent $S(\tau)$ as a linear combination of $E^{N,k}_{a,b}(\tau)$, up to a linear combination of 'constants' $\gamma_{a,b}(\tau)$, by using (the inverse of) the finite Fourier transform. Here we define the inverse finite Fourier transform $\widehat{f}$ of f by

$$\widehat{f}(n) = \frac{1}{N}\sum_{a \bmod N}\zeta_N^{-an} f(a).$$

It is known (and easily checked) that this transform satisfies

$$f(n) = \sum_{a \bmod N}\zeta_N^{an}\widehat{f}(a).$$

We will use this property to prove the following proposition.

Proposition 1. *For any two periodic functions satisfying*

$$f(-a)g(-b) = (-1)^k f(a)g(b) \textit{ for all integers } a, b,$$

The q-expansions of $S(\tau)$ and the Eisenstein series

$$\frac{1}{2}\sum_{a,b \bmod N} \widehat{f}(b)g(a)E_{a,b}^{N,k}$$

coincide. In other words, the identity

$$\sum_{n,m\geq 1} n^{k-1}f(n)g(m)q^{mn} = \frac{1}{2}\sum_{a,b \bmod N} \widehat{f}(b)g(a)(E_{a,b}^{N,k}(\tau) - \gamma_{a,b}^{N,k}(\tau)) \tag{14}$$

takes place.

Proof. We can assume without loss of generality that g and f not identically zero. From this follows that, by the imposed relation, there are n, m such that $g(m), g(-m), f(n), f(-n) \neq 0$. Then $f(-a) = (-1)^k \frac{g(-m)}{g(m)} f(a) = (-1)^k \frac{g(m)}{g(-m)} f(a)$, so $f(a) = f(-a)$ for all integers a or $f(a) = -f(-a)$ for all integers a. By symmetry the same property also holds for g. By writing out the definition of $\widehat{f}$ we find that

$$\widehat{f}(-a)g(-b) = (-1)^k \widehat{f}(a)g(b) \text{ for all integers } a, b.$$

Now we will use this property to prove the identity.

$$\begin{aligned}
\sum_{n,m\geq 1} f(n)g(m)n^{k-1}q^{mn} &= \sum_{a \bmod N} \sum_{\substack{m,n\geq 1 \\ m\equiv a \bmod N}} f(n)g(m)n^{k-1}q^{mn} \\
&= \sum_{a,b \bmod N} \sum_{\substack{m,n\geq 1 \\ m\equiv a \bmod N}} g(a)\widehat{f}(b)\zeta_N^{bn}n^{k-1}q^{mn} \\
&= \sum_{a,b \bmod N} \widehat{f}(b)g(a) \sum_{\substack{m,n\geq 1 \\ m\equiv a \bmod N}} \zeta_N^{bn}n^{k-1}q^{mn} \\
&= \frac{1}{2}\sum_{a,b \bmod N} \widehat{f}(b)g(a) \\
&\quad \times \left(\sum_{\substack{m,n\geq 1 \\ m\equiv a \bmod N}} \zeta_N^{bn}n^{k-1}q^{mn} + (-1)^k \sum_{\substack{m,n\geq 1 \\ m\equiv -a \bmod N}} \zeta_N^{-bn}n^{k-1}q^{mn} \right) \\
&= \frac{1}{2}\sum_{a,b \bmod N} \widehat{f}(b)g(a)(E_{a,b}^{N,k} - \gamma_{a,b}^{N,k}).
\end{aligned}$$

□

Remark. The proof did not actually use that k is a nonnegative integer. Therefore, this theorem is valid for general Eisenstein series of integral weight k.

7 The L-value at 4 for conductor 32

In [9], the L-values at 2 and 3 of an elliptic curve of conductor 32 are explicitly expressed as periods, and there is a general outline on how to derive such results. We will use this to compute a representation of the L-value at 4 of the elliptic curve as a period.

Recall that for a conductor 32 elliptic curve E, the L-series is known to coincide with that for the cusp form $f(\tau) = \eta_4^2\eta_8^2$. This will be shown to be a product of Eisenstein series.

We have the following (Lambert series) expansion:

$$\frac{\eta_8^4}{\eta_4^2} = \sum_{m\geq 1} \left(\frac{-4}{m}\right) \frac{q^m}{1-q^{2m}} = \sum_{m,n\geq 1} a(m)b(n)q^{mn},$$

where $a(m) := \left(\frac{-4}{m}\right)$ and $b(n) := n \bmod 2$ are as in [9]. This expansion can be obtained using Dirichlet convolution; in this case

$$a(m) = \sum_{\substack{n|m \\ \frac{m}{n} \text{ odd}}} c(n)\mu\left(\frac{m}{n}\right),$$

where $c(n)$ is the n-th term in the expansion on the left and μ is the Möbius function. Combining this with the identity

$$\eta_4^2\eta_8^2 = \frac{\eta_8^4}{\eta_4^2}\frac{\eta_4^4}{\eta_8^2}$$

and using the modular involution (2) we obtain

$$f(it) = \frac{1}{2t} \sum_{m_1,n_1\geq 1} a(m_1)b(n_1)e^{-2\pi m_1 n_1 t} \sum_{m_2,n_2\geq 1} a(m_2)b(n_2)e^{-2\pi m_2 n_2/(32t)}.$$

We apply this to the L-value at 4:

$$\begin{aligned} L(E,4) = L(f,4) &= \frac{1}{6}\int_0^1 f \log^3 q \frac{\mathrm{d}q}{q} = -\frac{(2\pi)^4}{6}\int_0^\infty f(it)t^3\,\mathrm{d}t \\ &= -\frac{(2\pi)^4}{2\cdot 6} \sum_{m_1,n_1,m_2,n_2\geq 1} a(m_1)b(n_1)b(m_2)a(n_2) \\ &\quad \times \int_0^\infty \exp\biggl(-2\pi\biggl(m_1n_1t + \frac{m_2n_2}{32t}\biggr)\biggr)t^2\,\mathrm{d}t. \end{aligned}$$

Performing the change of variable $t = \frac{n_2}{n_1}u$ yields

$$\begin{aligned} L(E,4) &= -\frac{4}{3}\pi^4 \sum_{m_1,n_1,m_2,n_2\geq 1} a(m_1)b(n_1)b(m_2)a(n_2)\frac{n_2^3}{n_1^3} \\ &\quad \times \int_0^\infty \exp\left(-2\pi\left(m_1n_2u + \frac{m_2n_1}{32u}\right)\right)u^2\,\mathrm{d}u \\ &= -\frac{4}{3}\pi^4 \int_0^\infty \sum_{m_1,n_2\geq 1} n_2^3 a(m_1)a(n_2)\exp(-2\pi m_1n_2u) \\ &\quad \times \sum_{m_2,n_1\geq 1} \frac{b(m_2)b(n_1)}{n_1^3}\exp\left(\frac{-2\pi m_2n_1}{32u}\right)u^2\,\mathrm{d}u. \end{aligned}$$

Now we perform another change of variable $v = \frac{1}{32u}$, then

$$u^2\,\mathrm{d}u = \frac{1}{(32v)^2} \times -\frac{\mathrm{d}v}{32v^2}.$$

We now have

$$\begin{aligned} L(E,4) &= -\frac{4\pi^4}{3\cdot 32^3}\int_0^\infty \sum_{m_1,n_2\geq 1} n_2^3 a(m_1)a(n_2)\exp\left(\frac{-2\pi m_1n_2}{32v}\right) \\ &\quad \times \sum_{m_2,n_1\geq 1}\frac{b(m_2)b(n_1)}{n_1^3}\exp(-2\pi m_2n_1v)\frac{\mathrm{d}v}{v^4} \\ &= -\frac{4\pi^4}{3\cdot 32^3}\int_0^\infty F_1(i/32v)(\delta^{-3}F_2)(iv)\frac{\mathrm{d}v}{v^4}, \end{aligned}$$

where

$$F_1(\tau) := \sum_{n,m\geq 1}\left(\frac{-4}{mn}\right)n^3q^{mn}$$

and

$$\begin{aligned} (\delta^{-3}F_2)(\tau) &= \sum_{m,n\geq 1}\frac{b(m)b(n)}{n^3}q^{mn} \\ &= \sum_{\substack{n\geq 1\\ n\text{ odd}}}\frac{q^n}{n^3(1-q^{2n})} \end{aligned}$$

We will now proceed by expressing $F_1(i/32\tau)$ as an eta quotient and $(\delta^{-3}F_2)(\tau)$ as a product of an eta quotient and a hypergeometric series in terms of an eta quotient. By PARI computations, we obtain

$$F_1(\tau) = \frac{\eta_4^{16}}{\eta_2^4\eta_8^4} - 32\frac{\eta_2^4\eta_8^{12}}{\eta_4^8},$$

thus

$$
\begin{aligned}
v^{-4}F_1(\tau)|_{\tau=i/(32v)} &= \left(2^{12}\frac{\eta_8^{16}}{\eta_4^4\eta_{16}^4} - 32\cdot 2^8\frac{\eta_4^{12}\eta_{16}^4}{\eta_8^8}\right)\Bigg|_{\tau=iv} \\
&= 2^{12}\left(\frac{\eta_8^{16}}{\eta_4^4\eta_{16}^4} - 2\frac{\eta_4^{12}\eta_{16}^4}{\eta_8^8}\right)\Bigg|_{\tau=iv} \\
&= -2^{12}F_1(2\tau)|_{\tau=iv}.
\end{aligned}
$$

If we define $\widetilde{x}(\tau) = 4\eta_4^{12}/\eta_2^4$ and then also the modular function

$$
X(\tau) = \tilde{x}(\tau)\cdot(1+\tilde{x}(\tau)^2)^{1/2} = \frac{4\eta_4^{12}}{\eta_2^{12}},
$$

then we have, by Duke's formula [1, eq. (2·6)], that

$$
\begin{aligned}
4(\delta^{-3}F_2)(\tau) &= \frac{X(\tau)\cdot {}_4F_3\left(\begin{matrix}1,\ 1,\ 1,\ 1\\ \frac{3}{2},\ \frac{3}{2},\ \frac{3}{2}\end{matrix}\ \middle|\ -4X(\tau)^2\right)}{{}_3F_2\left(\begin{matrix}\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\\ 1,\ 1\end{matrix}\ \middle|\ -4X(\tau)^2\right)} \\
&= \frac{X(\tau)\cdot H(-4X(\tau)^2)}{\Lambda(\tau)}.
\end{aligned}
$$

Clausen's formula [6, §2.5 eq.(2.5.7)] reads

$$
\Lambda(\tau) = {}_3F_2\left(\begin{matrix}\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\\ 1,\ 1\end{matrix}\ \middle|\ -4X(\tau)^2\right) = {}_2F_1\left(\begin{matrix}\frac{1}{2},\ \frac{1}{2}\\ 1\end{matrix}\ \middle|\ -\tilde{x}(\tau)^2\right)^2 = \frac{\eta_2^8}{\eta_4^4}.
$$

Now, using this, we can write

$$
\begin{aligned}
L(E,4) &= \frac{1}{6}\pi^4\int_0^\infty F_1(2\tau)(\delta^{-3}F_2)(\tau)|_{\tau=iv}\,\mathrm{d}v \\
&= \frac{1}{6}\pi^4\int_0^\infty \frac{1}{4}\left(\frac{\eta_8^{16}}{\eta_4^4\eta_{16}^4} - 32\frac{\eta_4^4\eta_{16}^{12}}{\eta_8^8}\right)\frac{\eta_4^{16}}{\eta_2^{20}}H(-4X(\tau)^2)|_{\tau=iv}\,\mathrm{d}v.
\end{aligned}
$$

If $x = \frac{4\eta_2^4\eta_8^8}{\eta_4^{12}}$, then

$$
\mathrm{d}x = \frac{4\eta_2^{12}\eta_8^8}{\eta_4^{16}}\frac{\mathrm{d}q}{q} = \frac{4\eta_2^{12}\eta_8^8}{\eta_4^{16}}2\pi i\,\mathrm{d}\tau.
$$

So by a change of variable, the differential form $2\pi i F_1(2iv)(\delta^{-3}F_2)(iv)\,\mathrm{d}v$ transforms into

$$
-\frac{1}{4}\left(\frac{\eta_8^{16}}{\eta_4^4\eta_{16}^4} - 32\frac{\eta_4^4\eta_{16}^{12}}{\eta_8^8}\right)\frac{\eta_4^{16}}{\eta_2^{20}}\times\frac{\eta_4^{16}}{\eta_2^{12}\eta_8^8}H(-4X^2)\,\mathrm{d}x.
$$

This is a rational function in $\left(\frac{\eta_4}{\eta_2}\right)^4$, $\left(\frac{\eta_8}{\eta_2}\right)^4$ and $\left(\frac{\eta_{16}}{\eta_2}\right)^4$ multiplied by $H(-4X^2)\,\mathrm{d}x$. Explicitly, we have the following:

$$\begin{aligned}2\pi i F_1(2\tau)(\delta^{-3}F_2)(\tau)\,\mathrm{d}\tau &= \frac{1}{4}\left(\frac{\eta_8^{16}}{\eta_4^4\eta_{16}^4} - 32\frac{\eta_4^4\eta_{16}^{12}}{\eta_8^8}\right)\frac{\eta_4^{32}}{\eta_2^{32}}\frac{1}{\eta_8^8} \times H(-4X^2)\,\mathrm{d}x \\ &= \frac{1}{4}\left(\left(\frac{\eta_8}{\eta_2}\right)^8\left(\frac{\eta_4}{\eta_2}\right)^{-4}\left(\frac{\eta_{16}}{\eta_2}\right)^{-4} - 32\left(\frac{\eta_8}{\eta_2}\right)^{-16}\left(\frac{\eta_4}{\eta_2}\right)^{4}\left(\frac{\eta_{16}}{\eta_2}\right)^{12}\right) \\ &\quad\times\left(\frac{\eta_4}{\eta_2}\right)^{32} \times H(-4X^2)\,\mathrm{d}x\end{aligned}$$

As in [9],

$$\left(\frac{\eta_8}{\eta_2}\right)^4 = \frac{\widetilde{x}}{4} = \frac{x}{4\sqrt{1-x^2}},$$

and we also have

$$\left(\frac{\eta_4}{\eta_2}\right)^4 = \left(\frac{x}{4(1-x^2)}\right)^{1/3}$$

and

$$\left(\frac{\eta_{16}}{\eta_2}\right)^4 = (1-\sqrt{1-x^2})\left(\frac{x}{2^{11}(1-x^2)^{7/4}}\right)^{1/3}.$$

Using these results we obtain

$$-2\pi F_1(2iv)(\delta^{-3}F_2)(iv)\,\mathrm{d}v = \frac{x^4 - 2(1-\sqrt{1-x^2})^4}{128(1-x^2)^{11/4}(1-\sqrt{1-x^2})} \times H(-4X^2)\,\mathrm{d}x.$$

If we express the algebraic relation in terms of $y = \sqrt{1-x^2}$, we get

$$\begin{aligned}-2\pi F_1(2iv)(\delta^{-3}F_2)(iv)\,\mathrm{d}v &= -\frac{(1-y^2)^2 - 2(1-y)^4}{128y^{9/2}(1-y)\sqrt{1-y^2}} \times H(-4X^2)\,\mathrm{d}y \\ &= \frac{(1-6y+y^2)\sqrt{1-y}}{128\sqrt{y^9(1+y)}}H(-4X^2)\end{aligned}$$

Since

$$X = \frac{x}{1-x^2} = \frac{\sqrt{1-y^2}}{y^2}$$

and $x(\tau)$ ranges from 0 to 1 when τ goes from $i\infty$ to 0, we have

$$L(E,4) = \frac{\pi^3}{1536}\int_0^1 \frac{x^4 - 2(1-\sqrt{1-x^2})^4}{(1-x^2)^{11/4}(1-\sqrt{1-x^2})} \times H\left(-4\frac{x^4}{(1-x^2)^2}\right)\mathrm{d}x$$

and

$$\begin{aligned}L(E,4) &= \frac{\pi^3}{1536}\int_0^1 \frac{(1-6y+y^2)\sqrt{1-y}}{\sqrt{y^9(1+y)}}H(-4X^2)\,\mathrm{d}y \\ &= \frac{\pi^3}{1536}\int_0^1 \frac{(1-6y+y^2)\sqrt{1-y}}{\sqrt{y(1+y)}}\iiint\limits_{[0,1]^3}\frac{\mathrm{d}y\,\mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}y_3}{y^4 + 4(1-y^2)(1-y_1^2)(1-y_2^2)(1-y_3^2)}.\end{aligned}$$

The following theorem summarises our findings in this section.

Theorem 2. *The L-value of an elliptic curve E of conductor 32 at 4 possesses the following expression as a period:*

$$L(E,4) = \frac{\pi^3}{1536} \int_{[0,1]^4} \cdots \int \frac{(1-6y+y^2)\sqrt{1-y}\,\mathrm{d}y\,\mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}y_3}{\sqrt{y(1+y)}\big(y^4+4(1-y^2)(1-y_1^2)(1-y_2^2)(1-y_3^2)\big)}.$$

8 General L-values for conductor 32

We will find an expression for $L(E,k)$ for both even and odd $k>1$ of the form

$$C\pi^k \int_0^\infty F_1(2iv)(\delta^{-k+1}F_2)(iv)\,\mathrm{d}v,$$

where C is an explicit rational constant and F_1 and F_2 are finite sums of Eisenstein series. In order to carry out this computation, we define the partial Fourier transform $\widetilde{E}_{d,b} = \widetilde{E}_{d,b}^{N,k}$ of an Eisenstein series:

$$\widetilde{E}_{d,b}^{N,k} = \sum_{a \bmod N} \zeta_N^{da} E_{a,b}^{N,k}.$$

Note that we also have

$$E_{a,b}^{N,k} = \frac{1}{N} \sum_{d \bmod N} \zeta_N^{-da} \widetilde{E}_{d,b}^{N,k}.$$

These functions have a simple series representation, which will help expressing series in terms of the Eisenstein series.

Proposition 2. *For any N, k, d and b the following holds:*

$$\widetilde{E}_{d,b}^{N,k} = \sum_{a \bmod N} \zeta_N^{da}\gamma_{a,b} + \sum_{n,m\geq 1} \zeta_N^{dm+bn} n^{k-1} q^{mn} + (-1)^k \sum_{n,m\geq 1} \zeta_N^{-dm-bn} n^{k-1} q^{mn} \qquad (15)$$

and for $k>1$, the first sum vanishes if $d \not\equiv 0 \bmod N$ and is otherwise equal to

$$(N-1)\gamma_{*,b} - \frac{\delta_{k,2}\cdot\beta_2\cdot 2\pi i}{N^3(\tau-\overline{\tau})} = (N-1)\beta_k \sideset{}{'}\sum_{\substack{m\in\mathbb{Z}\\ m\equiv b}} m^{-k} - \frac{\delta_{k,2}\cdot\beta_2\cdot 2\pi i}{N^2(\tau-\overline{\tau})}.$$

Note that for this reason the first sum vanishes for odd $k>1$.

Using (14) allows us to write S in the following way:

$$S(\tau) = \frac{1}{2} \sum_{a,b \bmod N} \widetilde{f}(a)\widetilde{g}(b)\widetilde{E}_{a,b}(\tau).$$

8.1 The case of even k

We have

$$\begin{aligned} L(E,k) &= -\frac{(2\pi)^k}{(k-1)!}\int_0^\infty f(it)t^{k-1}\,\mathrm{d}t \\ &= -\frac{(2\pi)^k}{2(k-1)!}\sum_{m_1,m_2,n_1,n_2\geq 1} a(m_1)b(n_1)b(m_2)a(n_2) \\ &\quad\times \int_0^\infty \exp\Bigg(-2\pi\bigg(m_1n_1t+\frac{m_2n_2}{32t}\bigg)\Bigg)t^{k-2}\,\mathrm{d}t, \end{aligned}$$

the change of variables $t=\frac{n_2}{n_1}u$ then gives

$$\begin{aligned} L(E,k) = -\frac{(2\pi)^k}{2(k-1)!}\int_0^\infty &\sum_{m_1,n_2\geq 1} n_2^{k-1}a(m_1)a(n_2)\exp(-2\pi m_1n_2u) \\ \times &\sum_{m_2,n_1\geq 1}\frac{b(m_2)b(n_1)}{n_1^{k-1}}\exp\left(\frac{-2\pi m_2n_1}{32u}\right)u^{k-2}\,\mathrm{d}u \end{aligned}$$

Now take $v=\frac{1}{32u}$ such that $u^{k-2}\,\mathrm{d}u=-\frac{1}{32^{k-1}}\frac{\mathrm{d}v}{v^m}$:

$$\begin{aligned} L(E,k) = -\frac{(2\pi)^k}{2\cdot 32^{k-1}(k-1)!} &\sum_{m_1,n_2\geq 1} n_2^{k-1}a(m_1)a(n_2)\exp\left(\frac{-2\pi m_1n_2}{32v}\right) \\ \times &\sum_{m_2,n_1\geq 1}\frac{b(m_2)b(n_1)}{n_1^{k-1}}\exp(-2\pi m_2n_1v)\frac{\mathrm{d}v}{v^k}. \end{aligned}$$

Define

$$\begin{aligned} F_1(\tau) &:= \sum_{m,n\geq 1} a(m)a(n)n^{k-1}q^{mn} = \sum_{m,n\geq 1}\left(\frac{-4}{mn}\right)n^{k-1}q^{mn}, \\ F_2(\tau) &:= \sum_{m,n\geq 1} b(m)b(n)n^{k-1}q^{mn} = \sum_{\substack{m,n\geq 1\\ m,n\text{ odd}}} n^{k-1}q^{mn}, \end{aligned}$$

so that

$$L(E,k) = -\frac{(2\pi)^k}{2\cdot 32^{k-1}(k-1)!}\int_0^\infty F_1\left(\frac{i}{32v}\right)\delta^{-k+1}(F_2)(iv)\frac{\mathrm{d}v}{v^k}.$$

It remains to write F_1 and F_2 in terms of Eisenstein series and to apply a modular transformation to F_1. As $\left(\frac{-4}{m}\right)=\frac{i^m-i^{-m}}{2i}$, we have

$$\left(\frac{-4}{mn}\right)=\frac{1}{4}(i^{m-n}+i^{-m+n}-i^{m+n}-i^{-m-n}),$$

so

$$F_1 = \frac{1}{4}\widetilde{E}^{4,k}_{1,-1} - \frac{1}{4}\widetilde{E}^{4,k}_{1,1} \\ = \frac{i}{2}(E^{4,k}_{1,-1} - E^{4,k}_{1,1}),$$

and as $\left(\frac{-4}{m}\right)^2 = m \bmod 2$, we have

$$b(n)b(m) = \frac{(-1)^{n+m} + (-1)^{-n-m} - (-1)^n - (-1)^{-n} - (-1)^m - (-1)^{-m} + 2}{8},$$

implying

$$F_2 = \frac{1}{8}(\widetilde{E}^{2,k}_{1,1} - \widetilde{E}^{2,k}_{1,0} - \widetilde{E}^{2,k}_{0,1} + \widetilde{E}^{2,k}_{0,0}) \\ = \frac{1}{4}(-E^{2,k}_{1,1} + E^{2,k}_{1,0}).$$

By (11) we have

$$F_1\left(\frac{i}{32v}\right) = \frac{i}{2}\left(E^{4,k}_{1,-1}\left(\frac{-1}{4^2(2iv)}\right) - E^{4,k}_{1,1}\left(\frac{-1}{4^2(2iv)}\right)\right) \\ = \frac{i}{2}(4\cdot 2iv)^k(-E^{4,k}_{1,-1}(2iv) + E^{4,k}_{1,1}(2iv)) \\ = -(8iv)^k F_1(2iv).$$

Thus we obtain

$$L(E,k) = \frac{-(2\pi)^k}{2\cdot 32^{k-1}(k-1)!}\int_0^\infty -(8iv)^k F_1(2iv)(\delta^{-k+1}F_2)(iv)\frac{\mathrm{d}v}{v^k} \\ = \frac{16(\pi i)^k}{2^k(k-1)!}\int_0^\infty F_1(2iv)(\delta^{-k+1}F_2)(iv)\,\mathrm{d}v.$$

8.2 The case of odd k

We have

$$L(E,k) = -\frac{(2\pi)^k}{(k-1)!}\int_0^\infty f(it)t^{k-1}\,\mathrm{d}t \\ = -\frac{(2\pi)^k}{2(k-1)!}\sum_{m_1,m_2,n_1,n_2\geq 1} a(n_1)b(m_1)b(m_2)a(n_2) \\ \times \int_0^\infty \exp\left(-2\pi\left(m_1n_1t + \frac{m_2n_2}{32t}\right)\right)t^{k-2}\,\mathrm{d}t,$$

the change of variables $t = \frac{n_2}{n_1} u$ then gives

$$L(E,k) = -\frac{(2\pi)^k}{2(k-1)!} \int_0^\infty \sum_{m_1,n_2\geq 1} n_2^{k-1} b(m_1)a(n_2)\exp(-2\pi m_1 n_2 u) \times \sum_{m_2,n_1\geq 1} \frac{b(m_2)a(n_1)}{n_1^{k-1}} \exp\left(\frac{-2\pi m_2 n_1}{32u}\right) u^{k-2}\,\mathrm{d}u.$$

Now take $v = \frac{1}{32u}$ such that $u^{k-2}\,\mathrm{d}u = -\frac{1}{32^{k-1}}\frac{\mathrm{d}v}{v^m}$:

$$L(E,k) = -\frac{(2\pi)^k}{2\cdot 32^{k-1}(k-1)!} \sum_{m_1,n_2\geq 1} n_2^{k-1} b(m_1)a(n_2)\exp\left(\frac{-2\pi m_1 n_2}{32v}\right) \times \sum_{m_2,n_1\geq 1} \frac{b(m_2)a(n_1)}{n_1^{k-1}} \exp(-2\pi m_2 n_1 v)\frac{\mathrm{d}v}{v^k}.$$

Define

$$\widehat{F}_1(\tau) := \sum_{m,n\geq 1} b(m)a(n)n^{k-1}q^{mn} = \sum_{\substack{m,n\geq 1\\ m \text{ odd}}} \left(\frac{-4}{n}\right) n^{k-1}q^{mn},$$

$$F_2(\tau) := \sum_{m,n\geq 1} b(m)a(n)m^{k-1}q^{mn} = \sum_{\substack{m,n\geq 1\\ m \text{ odd}}} \left(\frac{-4}{n}\right) m^{k-1}q^{mn},$$

so that

$$L(E,k) = -\frac{(2\pi)^k}{2\cdot 32^{k-1}(k-1)!} \int_0^\infty F_1\left(\frac{i}{32v}\right)\delta^{-k+1}(F_2)(iv)\frac{\mathrm{d}v}{v^k}.$$

Now it remains to write F_1 and F_2 in terms of Eisenstein series and to apply a modular transformation to F_1. Since

$$b(m)a(n) = \frac{i^n - i^{-n}}{2i}\cdot\frac{1-i^{2m}}{2} = \frac{i^n - i^{-n} - i^{n+2m} + i^{-n-2m}}{4i},$$

$$\begin{aligned}\widehat{F}_1 &= \frac{\widetilde{E}^{4,k}_{0,1} - \widetilde{E}^{4,k}_{2,1}}{4i}\\ &= \frac{E_{1,1} + E_{3,1}}{2i}\end{aligned}$$

and

$$\begin{aligned}F_2 &= \frac{\widetilde{E}^{4,k}_{1,0} - \widetilde{E}^{4,k}_{1,2}}{4i}\\ &= \frac{1}{4i}\sum_{a \bmod 4} i^{-a}(E^{4,k}_{a,2} - E^{4,k}_{a,0})\end{aligned}$$

By (11) we have

$$\begin{aligned}\widehat{F}_1\left(\frac{i}{32v}\right) &= \frac{i}{2}(E_{1,1}^{4,k}\left(\frac{-1}{4^2(2iv)}\right) - E_{3,1}^{4,k}\left(\frac{-1}{4^2(2iv)}\right)) \\ &= \frac{i}{2}(4\cdot 2iv)^k(-E_{1,-1}^{4,k}(2iv) + E_{1,-3}^{4,k}(2iv)) \\ &= -(8iv)^k F_1(2iv),\end{aligned}$$

where

$$F_1 = -E_{1,-1}^{4,k} + E_{1,-3}^{4,k}.$$

Thus, we obtain

$$\begin{aligned}L(E,k) &= \frac{-(2\pi)^k}{2\cdot 32^{k-1}(k-1)!}\int_0^\infty -(8iv)^k F_1(2iv)(\delta^{-k+1}F_2)(iv)\frac{\mathrm{d}v}{v^k} \\ &= \frac{16(\pi i)^k}{2^k(k-1)!}\int_0^\infty F_1(2iv)(\delta^{-k+1}F_2)(iv)\,\mathrm{d}v.\end{aligned}$$

We summarise our findings in this section as follows.

Theorem 3. *The L-value of an elliptic curve E of conductor 32 at $k \geq 2$ equals*

$$L(E,k) = \frac{16(\pi i)^k}{2^k(k-1)!}\int_0^\infty F_1(2iv)(\delta^{-k+1}F_2)(iv)\,\mathrm{d}v,$$

where F_1 and F_2 are Eisenstein series of weight k. Explicitly, for even k:

$$F_1 = \frac{i}{2}(E_{1,-1}^{4,k} - E_{1,1}^{4,k})$$

$$F_2 = \frac{1}{4}(E_{1,0}^{2,k} - E_{1,1}^{2,k}),$$

and for odd k:

$$F_1 = E_{1,-3}^{4,k} - E_{1,-1}^{4,k}$$

$$F_2 = \frac{i}{4}(\widetilde{E}_{1,2}^{4,k} - \widetilde{E}_{1,0}^{4,k}) = \frac{i}{4}\sum_{a \bmod 4} i^{-a}(E_{a,0}^{4,k} - E_{a,2}^{4,k}).$$

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